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ABSTRACT

The plane wave propagation in a half-space due to a uniformly distributed step load of pressure and shear on the surface was first studied by Bleich and Nelson. The material in the half-space was assumed to be elastic-ideally plastic. In this paper, we study the same problem for a general elastic-plastic meterial. The half-space can be initially prestressed. The results can be extended to the case in which the loads on the surface are not necessarily step loads, but with a restricted relation between the pressure and the shear stresses.

* Associate Professor of Applied Mechanics, Dept. of Materials Engineering, University of Illinois at Chicago Circle, Chicago, Illinois, 60680; Visiting at Stanford University, Dept. of Applied Mechanics, Stanford, Calif. 94305.

** Research Assistant, Dept. of Applied Mechanics, Stanford University, Stanford, Calif. 94305.

1. Introduction.

Several investigators have studied the propagation of elastic-plastic waves of combined stresses for various stress-strain relations and for various combinations of stress components [1-6]. For general initial and boundary value problems, analytical solutions are difficult to obtain except for a particular initial and boundary value problem [2] and/or a particular material property [1,6]. In [1], Bleich and Nelson obtained a closed from solution for the case of plane waves in an elastic-ideally plastic half-space subject to a step load of pressure and shear on the surface. In [2], Clifton studied the case of combined longitudinal and torsional waves in a thin-walled tube due to a step load of tension and torsion at the end of the tube. The materials considered by Clifton are general elastic-plastic materials with isotropic work-hardening property. In [6], Ting considered the case of two shear waves in an elastic, linearly work-hardening half-space subject to a series of step loadings and unloadings.

In the analyses of wave propagation in a thin-walled tube, the lateral inertia was ignored. While this is a good approximation, care must be exercised in comparing the theoretical analyses with an experimental result. For wave propogation in a half-space, the problem of lateral inertia does not arise. Thus the results of pressure-shear waves studied by Bleich and Nelson can be used without reservations for experimental verifications. Since an elastic-ideally plastic material is an idealization of real materials, the study of pressure-shears waves in a half-space of general elastic-plastic materials is desirable. This is presented in this paper.

The governing differential equations of the problem, the characteristics and the eigenvectors are presented in section 2. In section 3, we obtain the stress paths in the stress space which are used to obtain simple wave solutions. It is shown by an example that a simple wave solution can be a solution for the case in which the pressure and the shear applied at the boundary are arbitrary with a restricted relation between the pressure and the shear. The particular case in which the material is elastic-ideally plastic as considered in [1] is reduced in section 4, but presented in a form which can be used also when the half-space is initially prestressed.

2. The Basic Equations.

Let the half-space be bounded by the horizontal plane $x = 0$ of the cartesian coordinates (x,y,z) and extended to infinity on the side for which $x > 0$. Let $u(x,t)$ and $v(x,t)$ be the x and y components of the velocity of any particles which depend only on x and the time t. The z component of the velocity is assumed to be zero. The equations of motion for this plane motion is

$$\frac{\partial \sigma_1}{\partial x} = \rho \frac{\partial u}{\partial t} \qquad (1)$$

$$\frac{\partial \tau}{\partial x} = \rho \frac{\partial v}{\partial t} \qquad (2)$$

where ρ is the mass density of the half-space, and $\sigma_1 = \sigma_{xx}$, $\tau = \tau_{xy}$ for simplicity. The stress-strain relation for an isotropic work-hardening material is (see [7])

$$\frac{\partial \epsilon_{ij}}{\partial t} = \frac{1+\nu}{E} \frac{\partial \sigma_{ij}}{\partial t} - \frac{\nu}{E} \delta_{ij} \sigma_{kk} + \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial \lambda}{\partial t} \tag{3}$$

where E is Young's modulus, ν is Poisson's ratio and λ is a parameter which will be defined shortly. f is the yield condition which can be written as

$$f = \left(\frac{\sigma_1 - \sigma_2}{\theta}\right)^2 + \tau^2 = k^2 \tag{4}$$

where $\sigma_2 = \sigma_{yy} = \sigma_{zz}$, k is the yield stress, and θ is a constant which assumes the value $\sqrt{3}$ for the von Mises yield condition and the value 2 for the Tresca yield condition. Applying Eqs. (3) and (4) to the problem under consideration, and noticing that $\epsilon_{yy} = \epsilon_{zz} = 0$ and

$$\frac{\partial u}{\partial x} = \frac{\partial \epsilon_{xx}}{\partial t}, \quad \frac{1}{2} \frac{\partial v}{\partial x} = \frac{\partial \epsilon_{xy}}{\partial t} \tag{5}$$

by the continuity requirement, we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{E} \frac{\partial \sigma_1}{\partial t} - \frac{2\nu}{E} \frac{\partial \sigma_2}{\partial t} + s \frac{\partial \lambda}{\partial t} \tag{6}$$

$$0 = - \frac{2\nu}{E} \frac{\partial \sigma_1}{\partial t} + \frac{2(1-\nu)}{E} \frac{\partial \sigma_2}{\partial t} - s \frac{\partial \lambda}{\partial t} \tag{7}$$

$$\frac{\partial v}{\partial x} = \frac{1}{\mu} \frac{\partial \tau}{\partial t} + 2\tau \frac{\partial \lambda}{\partial t} \tag{8}$$

where μ is the shear modulus and

$$s = \frac{2}{\theta^2} (\sigma_1 - \sigma_2). \tag{9}$$

λ in Eq.(3) can be expressed in terms of f as (see [7]),

$$\frac{\partial\lambda}{\partial t} = \frac{\theta^2\alpha(k)}{4k^2E}\frac{\partial f}{\partial\sigma_{k\ell}}\frac{\partial\sigma_{k\ell}}{\partial t} \tag{10}$$

where $\alpha(k)$ characterizes the work-hardening property which can be determined from the stress-strain curve for a simple tension test. If $E_p(\sigma)$ is the slope of the stress-strain curve for a simple tension test expressed in terms of the tensile stress σ, we have (see [2]),

$$\alpha(k) = \frac{E}{E_p(\theta k)} - 1 \tag{11}$$

In the elastic region, $E_p = E$ and $\alpha = 0$, while in the ideally plastic region, $E_p = 0$ and $\alpha = \infty$. We shall assume that E_p is a monotonically decreasing function of k. Then, by Eq.(11), $\alpha(k)$ is an increasing function of k.

Equations (1), (2), (6), (7), (8), and (10) can be written in a matrix form:

$$\underset{\sim}{A}\,\underset{\sim}{w}_t + \underset{\sim}{B}\,\underset{\sim}{w}_x = 0 \tag{12}$$

where the subscripts x and t denote partial differentiation with respect to these variables and

$$\underset{\sim}{A} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{E} & -\frac{2\nu}{E} & 0 & s \\ 0 & 0 & -\frac{2\nu}{E} & \frac{2(1-\nu)}{E} & 0 & -s \\ 0 & 0 & 0 & 0 & \frac{1}{\mu} & 2\tau \\ 0 & 0 & s & -s & 2\tau & \frac{4k^2E}{\theta^2\alpha} \end{bmatrix}$$

$$
\underset{\sim}{B} = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \underset{\sim}{W} = \begin{bmatrix} u \\ v \\ \sigma_1 \\ \sigma_2 \\ \tau \\ \lambda \end{bmatrix}
$$

Notice that both matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are symmetric. By letting $\alpha \to \infty$, we obtain alternate equations for elastic-ideally plastic materials considered in [1]. The characteristics c of Eq.(12) are the roots of the determinant $|c\underset{\sim}{A}-\underset{\sim}{B}| = 0$, (see [8]). After expanding the determinant, we obtain:

$$
c^2 D(c) = 0 \tag{13}
$$

where

$$
\begin{aligned}
D(c) = {} & \left(\frac{c}{c_2}\right)^4 \left\{\frac{3s^2}{\beta} + \frac{4\tau^2}{\beta} + \frac{12k^2}{\theta^2\alpha(\beta+1)}\right\} \\
& -\left(\frac{c}{c_2}\right)^2 \left\{\left(1+\frac{3}{\beta}\right)s^2 + \frac{4(\beta+4)}{\beta^2}\,\tau^2 + \frac{4k^2(\beta+7)}{\theta^2\alpha(\beta+1)}\right\} + \left\{s^2 + \frac{4k^2(\beta+4)}{\theta^2\alpha(\beta+1)}\right\}
\end{aligned} \tag{14a}
$$

$c_2^2 = \mu/\rho$ is the shear wave speed and β is the parameter introduced by Bleich and Nelson which is related to Poisson's ratio ν by the equation

$$
\beta = \frac{2(1+\nu)}{1-2\nu} \tag{15}
$$

As ν assumes the range $(0, \frac{1}{2})$, β has the range $(2,\infty)$. From Eq. (13), we have either $c = 0$ or c obtained by the roots of $D(c) = 0$.

To see the positions of the roots of $D(c) = 0$, we rewrite Eq.(14a) in the form:

$$D(c) = s^2\left(\frac{c^2}{c_2^2} - 1\right)\left(\frac{3}{\beta}\frac{c^2}{c_2^2} - 1\right) + \frac{4\tau^2}{\beta}\frac{c^2}{c_2^2}\left(\frac{c^2}{c_2^2} - \frac{c_1^2}{c_2^2}\right)$$
$$+ \frac{12k^2}{\theta^2\alpha(\beta+1)}\left(\frac{c^2}{c_1^2} - 1\right)\left(\frac{c^2}{c_2^2} - \frac{c_1^2}{c_2^2}\right) \qquad (14b)$$

where c_1 is the dilational wave speed and has the value

$$\frac{c_1^2}{c_2^2} = \frac{\beta+4}{3} \qquad (16)$$

From Eq.(14b), it is easily seen that $D(0) \geq 0$, $D(c_2) \leq 0$ and $D(c_1) \geq 0$. Therefore, if $\pm c_f$ $\pm c_s$ denote the roots of $D(c) = 0$, we have the relation:

$$0 \leq c_s \leq c_2 \leq c_f \leq c_1 \qquad (17)$$

c_f and c_s correspond, respectively, to the fast wave speed and the slow wave speed.

Two extreme cases can be reduced from Eq.(14b). In the elastic region, $\alpha = 0$ and $D(c) = 0$ gives $c = c_1$ and $c = c_2$. In the ideally plastic region, $\alpha = \infty$ and the roots of Eq.(14b) reduce to the ones obtained in [1].

Other extreme cases can be reduced from Eq.(14a). We simply list the results in the following:

(i) When $s = 0$,

$$c_f = c_1, \quad 0 \leq c_s \leq c_2 \qquad (18)$$

(ii) When $\tau = 0$, and $\beta \geq 3$,

$$\sqrt{\frac{\beta}{3}}\, c_2 \leq c_f \leq c_1 \,, \quad c_s = c_2 \tag{19}$$

(iii) When $\tau = 0$, and $\beta < 3$

$$c_2 \leq c_f \leq c_1 \,, \quad c_s = c_2 \quad \text{for} \quad s \leq s^* \tag{20a}$$

$$c_f = c_2 \,, \sqrt{\frac{\beta}{3}}\, c_2 \leq c_s \leq c_2 \quad \text{for} \quad s \geq s^* \tag{20b}$$

where s^* is determined by

$$\alpha\left(\frac{2s^*}{\theta}\right) = \frac{\beta}{3-\beta} \,. \tag{21}$$

Physically, s^* is the value at which the plane plastic compressional wave speed has the same value as c_2, the elastic shear wave speed.

Both c_f and c_s are functions of s ant τ. If we introduce a new variable φ by

$$\tau = k \sin \varphi \tag{22a}$$

then Eqs. (4) and (9) give

$$s = \frac{2}{\theta} k \cos \varphi \,. \tag{22b}$$

With this change of variables, c_f and c_s become functions of k and φ. In the $s \sim \tau$ plane, a constant k gives a yield surface while a constant φ gives a straignt line through the origin. Regarding c_f and c_s as functions of k and φ, it can be shown that, for $0 < \varphi < \frac{\pi}{2}$,

$$\frac{\partial c_f}{\partial \varphi} > 0 \ , \quad \frac{\partial c_s}{\partial \varphi} < 0 \ , \quad \frac{\partial c_f}{\partial k} < 0 \ , \quad \frac{\partial c_s}{\partial k} < 0 \ . \tag{23}$$

From Eq.(23), and the information presented in Eqs.(18) to (20), c_f and c_s have the ranges:

$$0 \leq c_s \leq c_2 \ , \quad \sqrt{\frac{\beta}{3}}\, c_2 \leq c_f \leq c_1 \quad \text{for} \quad \beta \geq 3 \tag{24a}$$

$$0 \leq c_s \leq c_2 \ , \quad c_2 \leq c_f \leq c_1 \quad \text{for} \quad \beta < 3 \tag{24b}$$

The characteristic condition along a characteristics c is obtained by [8],

$$\underset{\sim}{\ell}^T \underset{\approx}{A} \frac{d\underset{\sim}{w}}{dt} = 0 \tag{25}$$

where $\frac{d\underset{\sim}{w}}{dt} = \underset{\sim}{w}_x c + \underset{\sim}{w}_t$ is the total derivative along the characteristics c and $\underset{\sim}{\ell}^T$ is the transpose of the left eigenvector $\underset{\sim}{\ell}$ which is obtained by the equation

$$\underset{\sim}{\ell}^T (c\underset{\approx}{A} - \underset{\approx}{B}) = 0 \ . \tag{26a}$$

In particular, for the characteristics $c = 0$, Eq.(25) reduces to Eqs. (7) and (10).

When $\underset{\approx}{A}$ and $\underset{\approx}{B}$ are symmetric as in the present case, Eq.(26a) can be written as

$$(c\underset{\approx}{A} - \underset{\approx}{B})\underset{\sim}{\ell} = 0 \ . \tag{26b}$$

Thus the left eigenvector and the right eigenvector are identical. For $c = \pm c_f$ or $\pm c_s$, $\underset{\sim}{\ell}$ of Eq.(26b) has the solution

$$\underset{\sim}{\ell} = \begin{bmatrix} \Psi \\ 1 \\ -\rho c \Psi \\ \Phi \\ -\rho c \\ \Theta \end{bmatrix} \tag{27}$$

where

$$\Psi = -\frac{\tau}{s}\frac{c_1^2}{c_2^2} - \frac{s\beta}{4\tau}\left(1 + \frac{\beta+4}{\beta+1}\frac{4k^2}{\theta^2 s^2 \alpha}\right)\left(1 - \frac{c_2^2}{c^2}\right) \tag{28a}$$

$$\Phi = \frac{(\beta-2)}{3}\frac{\rho c \tau}{s} + \frac{\rho c s \beta}{4\tau}\left(1 \div \frac{\beta-2}{\beta+1}\frac{4k^2}{\theta^2 s^2 \alpha}\right)\left(1 - \frac{c_2^2}{c^2}\right) \tag{28b}$$

$$\Theta = \frac{1}{2\tau c}\left(\frac{c^2}{c_2^2} - 1\right)$$

Ψ and Φ can be writeen in simpler forms by eliminating α from the equation $D(c) = 0$ where $D(c)$ is expressed in Eq.(14b):

$$\Psi = \frac{s}{\tau}\frac{c^2 - c_2^2}{c^2 - c_1^2} \tag{29a}$$

$$\Phi = \frac{\rho s (c^2 - c_2^2)(c^2 - \beta c_2^2)}{2\tau c (c^2 - c_1^2)} \,. \tag{29b}$$

3. Generalized Simple Wave Solutions.

Simple wave solutions are particular solutions of Eq.(12) in which $\underset{\sim}{w}$ is a constant vector along a characteristics [9]. Thus, if $\underset{\sim}{w}$ is a constant vector along the c_f we have a fast simple wave solution.

If $\underset{\sim}{w}$ is a constant along the c_s we have a slow simple wave solution [2]. According to the theory of generalized simple waves developed in [9], in the region where the simple wave solution is valid, $d\underset{\sim}{w}$ is proportional to the eigenvector $\underset{\sim}{\ell}$. Therefore, by Eq.(27),

$$\frac{du}{\Psi} = \frac{dv}{1} = \frac{d\sigma_1}{-\rho c \Psi} = \frac{d\sigma_2}{\Phi} = \frac{d\tau}{-\rho c} = \frac{d\lambda}{\Theta} \tag{30}$$

Equation (30) is equivalent to five differential equations of the first order. Since Ψ, Φ and Θ are functions of the stresses only, these five equations are not all coupled. In particular, since

$$\frac{d\sigma_1 - d\sigma_2}{-\rho c \Psi - \Phi} = \frac{d\tau}{-\rho c} ,$$

we have, by Eqs.(9) and (28),

$$\frac{ds}{d\tau} = -\frac{4\tau}{\theta^2 s} - \frac{12\beta k^2}{\theta^4 \tau s(\beta+1)\alpha}\left(1 - \frac{c_2^2}{c^2}\right) , \tag{31a}$$

or, by Eqs.(9) and (29),

$$\frac{ds}{d\tau} = \frac{s(c^2-c_2^2)(3c^2-\beta c_2^2)}{\theta^2 \tau c^2 (c^2-c_1^2)} . \tag{31b}$$

The right hand side of Eq.(31a) or Eq.(31b) is a function of s and τ only. Hence Eq.(31a), or Eq.(31b), is a differential equation by itself. To obtain the stress paths in the $(\sigma_1, \sigma_2, \tau)$ space for simple wave solutions, we need, in addition to Eq.(31b), another equation which can be written, by Eqs.(30) and (29a),

$$\frac{d\sigma_1}{d\tau} = \frac{s}{\tau} \frac{c^2-c_2^2}{c^2-c_1^2} . \tag{32}$$

Equations (31b) and (32), when integrated, given two-parameter family of space curves(for each c_f and c_s) in the three-dimensional space $(\sigma_1, \sigma_2, \tau)$. These space curves can be constructed if we have their projections on any two planes which are not parallel such as the $\sigma_1 \sim \tau$ and the $\sigma_2 \sim \tau$ planes. In view of Eqs.(31b) and (32), we will use the $s \sim \tau$ plane and the $\sigma_1 \sim \tau$ plane instead. The reason is obvious. Equation (31b), when integrated, gives only one-parameter family of curves in the $s \sim \tau$ plane. As to the projected curves on the $\sigma_1 \sim \tau$ plane, we also need only one-parameter family of curves. The second parameter family of curves can be obtained by translating the first parameter family of curves in the σ_1~direction. This is clear from Eqs. (31b) and (32).

Before we illustrate how to construct these curves (or stress paths), for simple wave solutions, we will present some properties of these curves in the $s \sim \tau$ plane and the $\sigma_1 \sim \tau$ plane. First, consider these curves in the $\sigma_1 \sim \tau$ plane. By Eqs. (32) and (17), we have

$$\left(\frac{d\sigma_1}{d\tau}\right)_{c=c_f} < 0$$

$$\left(\frac{d\sigma_1}{d\tau}\right)_{c=c_s} > 0 \qquad (33)$$

Using Eqs. (32) and (14b), it can be shown that

$$\left(\frac{d\sigma_1}{d\tau}\right)_{c=c_f} \cdot \left(\frac{d\sigma_1}{d\tau}\right)_{c=c_s} = -1 . \qquad (34)$$

In other words, the stress paths for the fast simple waves and the slow simple waves are orthogonal to each other in the $\sigma_1 \sim \tau$ plane. It

should be noticed that they are orthogonal in the $\sigma_1 \sim \tau$ plane provided they intersect each other in the $(\sigma_1, \sigma_2, \tau)$ space. If a stress path for the fast simple wave and a stress path for the slow simple wave do not intersect in the $(\sigma_1, \sigma_2, \tau)$ space, their projections in the $\sigma_1 \sim \tau$ plane are not necessarily orthogonal to each other.

Next, consider the projections of the stress paths in the $s \sim \tau$ plane. From Eqs. (24) and (31a), we have

$$\left(\frac{ds}{d\tau}\right)_{c=c_f} \leq -\frac{4\tau}{\theta^2 s} , \quad \text{for all } \beta . \tag{35}$$

For slow waves, $ds/d\tau$ depends on whether c_s^2/c_2^2 is larger or smaller than $\beta/3$. (If $\beta > 3$, c_s^2/c_2^2 is automatically smaller than $\beta/3$.) By Eqs. (24), (31a), and (31b), we have

$$-\frac{4\tau}{\theta^2 s} \leq \left(\frac{ds}{d\tau}\right)_{c=c_s} \leq 0 , \quad \text{for } \frac{c_s^2}{c_2^2} < \frac{\beta}{3} . \tag{36a}$$

If $\frac{c_s^2}{c_2^2} \geq \frac{\beta}{3}$, (this can happen only when $\beta \leq 3$) it is seen by Eqs. (24b) and (31b) that

$$0 \leq \left(\frac{ds}{d\tau}\right)_{c=c_s} \leq \frac{s}{\tau} , \quad \text{for } \frac{c_s^2}{c_2^2} \geq \frac{\beta}{3} \tag{36b}$$

In Figs. 1 and 2, we give two examples of stress paths projections in the $\sigma_1 \sim \tau$ plane and the $s \sim \tau$ plane by integrating Eqs. (31b) and (32) numerically. s in Figs. 1(b) and 2(b) has been changed to $(\sigma_1 - \sigma_2)$ according to Eq. (9). k_o is the initial yield stress. In Fig. 1, $\beta = 5$, which shows a typical example of $\beta > 3$ while in Fig. 2, the value $\beta = 2.25$ gives an example of $\beta < 3$. In both cases, the

von Mises criterion is used and hence $\theta^2 = 3$. The work-hardening property $\alpha(k)$ are chosen differently in Figs. 1 and 2. They are chosen for illustrative purpose and do not necessarily represent a real material. In both figures, the solid lines marked by $s_1, s_2, \ldots,$ are the stress paths for the slow simple waves while the dashed lines marked by $f_1, f_2, \ldots,$ are the stress paths for the fast simple waves. The subscripts $1, 2, \ldots,$ have no particular meaning except to identify the stress paths in the $(\sigma_1, \sigma_2, \tau)$ space. For instance, the curve s_3 in Fig. 1(a) and the curve s_3 in Fig. 1(b) represent the same space curve in the $(\sigma_1, \sigma_2, \tau)$ space. In Fig. 2(b) the point d on the s-axis corresponds to the value s^* defined in Eq. (21). In fact, $s = s^*$, $\tau = 0$ is a singularity of Eqs. (31b) and (32). The dotted line in Fig. 2(b) is the locus of $c_s^2/c_2^2 = \beta/3$. It is seen that Eqs. (33)-(36) are satisfied by the results obtained in Figs. 1 and 2.

The arrow heads in the figures show the directions along which the stress state should be changed to insure a continuous plastic flow. It is apparent, by Eq. (23) and Eqs. (33)-(36), that along these directions shown by the arrow heads, the wave speeds c_f and c_s decrease on the stress paths for the fast simple waves and the slow simple waves respectively. This is an important requirement for constructing a simple wave solution.

In integrating the differential equations, Eqs. (31b) and (32) initial conditions are required. The initial conditions for the results obtained in Figs. 1 and 2 are chosen in such a way that Figs. 1 and 2 can be used to construct simple wave solutions for the case in which the half-space is initially stress free. No new calculations are required

if the half-space is initially pre-stressed. By rearranging the curves in Figs. 1(b) and 2(b), Figs. 1 and 2 can be used for the case in which the half-space is initially pre-stressed. We will illustrate this later in this section.

When the half-space is initially stress free, the disturbance caused by a load on the surface of the half-space will propagate at the elastic wave speed c_1 or c_2. The response at the wave front, and possibly a finite region behind the wave front, is elastic and Eqs. (1), (2), (6), (7), and (8) apply if we let $\partial\lambda/\partial t = 0$. In particular, Eq. (7) yields

$$\frac{d\sigma_2}{d\sigma_1} = \frac{\nu}{1-\nu}$$

or, since σ_1 and σ_2 are initially zero, we have, making use of Eq. (15),

$$\sigma_2 = \frac{\beta-2}{\beta+4}\,\sigma_1 \ . \tag{37}$$

Therefore, the initial yield limit k_o is reached when, by Eq. (4),

$$\left(\frac{\sigma_1-\sigma_2}{\theta}\right)^2 + \tau^2 = k_o^2 \ . \tag{38}$$

Equation (38) is the initial yield surface shown in Figs. 1(b) and 2(b) while elimination of σ_2 between Eqs. (37) and (38) gives the initial surface shown in Figs. 1(a) and 2(a). In other words, Eqs. (37) and (38) represent a curve in the $(\sigma_1, \sigma_2, \tau)$ space and the projections of this curve in the $s \sim \tau$ plane and the $\sigma_1 \sim \tau$ plane are denoted by initial yield surface in Figs. 1 and 2. The initial conditions, or the

"starting points", for the stress paths for the fast simple waves shown in Figs. 1 and 2 are taken on the τ-axis and on the curve represented by Eqs. (37) and (38). The starting points of the stress paths for the slow simple waves are taken on the curve represented by Eqs. (37) and (38), and on a curve on the $\sigma_1 \sim \sigma_2$ plane in the $(\sigma_1, \sigma_2, \tau)$ space which we will derive next. This curve is in fact a stress path for the fast simple waves which lies on the $\sigma_1 \sim \sigma_2$ plane.

From Eqs. (31b) and (32) we have, for $c = c_f$,

$$\frac{ds}{d\sigma_1} = \frac{3}{\theta^2} - \frac{\beta}{\theta^2} \frac{c_f^2}{c_2^2} . \tag{39}$$

When $\tau = 0$, $\frac{\theta}{2} s = k$ and, after solving c_f^2/c_2^2 from Eq. (13) and substituting the result into Eq. (39), we obtain

$$\frac{\theta^2}{2} \frac{ds}{d\sigma_1} = \frac{6}{(\beta+1)\alpha(\theta s/2)+(\beta+4)} \quad \text{for} \quad \tau = 0 . \tag{40a}$$

This is the differential equation for the stress paths for the fast simple waves when $\tau = 0$. Since $\alpha(k)$ is a given function of k, $\alpha(\theta s/2)$ is a known function of s and Eq. (40a) can be integrated. The initial condition is taken from the intersection of the curve represented by Eqs. (37) and (38) with the plane $\tau = 0$, which gives:

$$s = \frac{2}{\theta} k_o , \qquad \sigma_1 = \frac{\theta}{6} (\beta+4)k_o . \tag{40b}$$

Equations (40a) and (40b) represent a curve on the plane $\tau = 0$. The starting points of the stress paths for the slow simple waves shown in Figs. 1 and 2 are taken on the curve represented by Eqs. (37) and (38) and the curve represented by Eqs. (40a) and (40b).

Now, we are in the position to construct simple wave solutions by using the stress paths obtained in Figs. 1 and 2 for the case when the half-space is initially stress free. The stress paths shown in Figs. 1 and 2 are for a plastic region. In an elastic region, the stress paths corresponding to the fast simple waves $c = c_1$ are horizontal lines parallel to the σ_1 and s-axes while the stress paths corresponding to the slow simple waves $c = c_2$ are vertical lines parallel to the τ-axis. The stress paths for an elastic region are not oriented, i.e., the stress state can be changed in either direction. We define an "admissible stress path" by the path which consists of one or more stress paths in the elastic and/or plastic regions in such a way that when one moves along the path, the wave speeds are non-increasing. Thus, for instance, the paths oaij, obg and ode in Fig. 2 are admissible stress paths. For each admissible stress path, a simple wave solution can be constructed. For instance, a simple wave solution for the admissible stress path obg is shown in Fig. 3(a) where the solution corresponds to that of a step load of $\sigma_1 = \sigma_1^g$, $\tau = \tau^g$ on the surface of the half-space. A superscript g denote the value at the point g in Fig. 2. Since the characteristics c are functions of the stresses σ_1, σ_2, τ, and since the stresses are constant along the characteristics for a simple wave solution, the characteristics are straight lines as shown in Fig. 3(a). Each point on the stress path obg corresponds to a characteristic line in the $x \sim t$ plane. The position of the point on obg determines the stresses along the characteristics and also the slope of the characteristics. Thus σ_1 and τ are determined from Fig. 2(a) and σ_2 is determined from Fig. 2(b) accordingly.

Fig. 3(b) shows another simple wave solution corresponding to the same stress path obg. The loads on the surface, $\sigma_1(0,t)$ and $\tau(0,t)$ are not step loads. $\sigma_1(0,t)$ and $\tau(0,t)$ however, should be prescribed in such a way that, as t increases, $\sigma_1(0,t)$ and $\tau(0,t)$ follow the stress path obg. Thus, one of them, say $\sigma_1(0,t)$, can be prescribed almost arbitrarily, and the other one $\tau(0,t)$ is obtained by $\sigma_1(0,t)$ and the stress path obg. It should be noted that both $\sigma_1(0,t)$ and $\tau(0,t)$ do not have to be continuous functions of t as shown in Fig. 3(b).

When the half-space is prestressed, the stress paths in the $s \sim \tau$ plane remain unchanged while the stress paths in the $\sigma_1 \sim \tau$ plane need some modifications. To illustrate this, let us consider the case in which the half-space is pre-sheared. In Fig. 1, suppose that the pre-sheared stress is at the point a. An admissible stress path in Fig. 1(b) is abdegh which consists of the stress paths f_2 and s_4. The corresponding stress path in Fig. 1(a) however, is not ab'd'e'g'h' since g in Fig. 1(b) and g' in Fig. 1(a) do not represent the same point in the (σ_1,σ_2,τ) space. This can be verified easily as the ordinates of g' and g are not the same. Similarly, b', d', e' in Fig. 1(a) and b, d, e in Fig. 1(b) do not represent the same points in the (σ_1,σ_2,τ) space. As we explained earlier, the stress paths obtained in the $\sigma_1 \sim \tau$ plane can be used for other initial conditions if we translate the curves in the horizontal direction. Thus we translate the curves s_1, s_2, s_3, and s_4 in Fig. 1(a) until their intersections with f_2 give the same ordinates as b, d, e and g respectively of Fig. 1(b). The result of this translation is shown in

Fig. 4. Now, abdegh in Fig. 1(b) and Fig. 4 represent a continuous curve in the $(\sigma_1, \sigma_2, \tau)$ space, so do abdi and abdegj. They are all admissible stress paths. For the stress path abdegj, the last segment gj corresponds to an elastic unloading. Once we obtain an admissible stress path, a simple wave solution can be constructed as illustrated in the previous example, and Figs. 3.

The case when the half-space is pre-compressed can be analysed in a similar manner. If the half-space is pre-sheared and pre-compressed, then, in addition to the initial values of σ_1 and τ, the initial value of σ_2 has to be specified. This is so because σ_2 depends on how the half-space is pre-sheared and pre-compressed. Again, Figs. 1 and 2 can be used for this case with minor modifications.

4. Ideal Elastic-Plastic Materials.

When the material is elastic-ideally plastic, i.e., $\alpha = \infty$, the analysis is greatly simplified. Instead of projecting stress paths on the $s \sim \tau$ plane and the $\sigma_1 \sim \tau$ plane, only the latter is required. Moreover, only one stress path for the fast simple waves and the slow simple waves need to be calculated; the rest are obtained by a translation of the curve calculated. No modifications are necessary for the case when the half-space is pre-stressed.

For an ideally elastic-plastic material, the yield condition Eq. (4) is replaced by Eq. (38) where k_o is a fixed constant. Thus σ_2 is no longer an independent unknown but can be expressed in terms of σ_1 and τ. The characteristic equation $D(c) = 0$ now becomes, by Eqs. (14b) and (38),

$$(k_o^2-\tau^2)(c^2-c_2^2)(3c^2-\beta c_2^2) + \theta^2\tau^2c^2(c^2-c_1^2) = 0 \ . \tag{41}$$

Therefore, c does not depend on σ_1. The values of c when $\tau = 0$ and $\tau = k_o$ can be obtained easily from Eq. (41). The result is

$$c_f = \sqrt{\frac{\beta}{3}}\, c_2 \ , \quad c_s = c_2 \ , \quad \text{when} \quad \tau = 0 \quad \text{and} \quad \beta \geq 3 \tag{42a}$$

$$c_f = c_2 \ , \quad c_s = \sqrt{\frac{\beta}{3}}\, c_2 \ , \quad \text{when} \quad \tau = 0 \quad \text{and} \quad \beta \leq 3 \tag{42b}$$

$$c_f = c_1 \ , \quad c_s = 0 \ , \quad \text{when} \quad \tau = k_o \ . \tag{42c}$$

It can be shown that

$$\frac{\partial c_f}{\partial \tau} < 0 \ , \quad \frac{\partial c_s}{\partial \tau} > 0 \ . \tag{43}$$

Hence, by Eqs. (42) and (43), c_f and c_s have the ranges

$$0 < c_s \leq c_2 \ , \quad \sqrt{\frac{\beta}{3}}\, c_2 \leq c_f \leq c_1 \quad \text{for} \quad \beta \geq 3 \tag{44a}$$

$$0 \leq c_s \leq \sqrt{\frac{\beta}{3}}\, c_2 \ , \quad c_2 \leq c_f \leq c_1 \quad \text{for} \quad \beta < 3. \tag{44b}$$

This agrees with the result obtained in [1].

Using Eqs. (38) and (41), the right hand side of Eq. (32) can be written in terms of σ_1 and τ alone. The result is,

$$\frac{d\sigma_1}{d\tau} = -\frac{Q \pm \sqrt{Q^2+16\theta^2\tau^2(k_o^2-\tau^2)}}{4\theta\tau\sqrt{k_o^2-\tau^2}}$$

where

$$Q = (\beta-3)k_o^2 + \{\frac{\beta+4}{3}\,\theta^2 - (\beta-3)\}\tau^2$$

The $+$ sign in Eq. (45) is for the fast simple waves while the $-$ sign is for the slow simple waves. Equation (45) can be integrated in a closed form in terms of elliptic integrals. The particular case in which $\theta^2 = 3$ was obtained in [1]. In Fig. 5, we show the results of an integration of Eq. (45) for three cases: $\beta > 3$, $\beta = 3$, and $\beta < 3$. In each case only one curve for the fast simple waves and the slow simple waves needs to be calculated. The rest of the curves are obtained by translation in the σ_1-direction of the curve calculated. Again, the solid lines with arrow heads are the stress paths for the slow simple waves while the dashed lines with arrow heads are for the fast simple waves. In addition, the σ_1~axis is also an admissible stress path which corresponds to a constant wave speed of $\sqrt{\beta/3}\, c_2$. Notice that Eq. (34) still applies and therefore the stress paths for the fast simple waves and the slow simple waves are orthogonal to each other everywhere. Notice also the angle at which the stress paths intersect the σ_1~axis. The stress paths for the slow simple waves shown by solid lines intersect the σ_1~axis at 90° for $\beta > 5$, at 45° for $\beta = 3$, but never intersect the σ_1~axis for $\beta < 3$. On the other hand, the stress paths for the fast simple waves shown by dashed lines never intersect the σ_1~axis for $\beta > 5$ but intersect the σ_1~axis at 45° for $\beta = 3$ and at 90° for $\beta < 3$. The stress paths for fast and slow simple waves do intersect the horizontal line $\tau = k_o$; one intersects at 90° and the other is tangent to the line $\tau = k_o$.

Fig. 5 can be used for obtaining simple wave solutions regardless of whether the half-space is initially stress free or not. Any point in the $\sigma_1 \sim \tau$ plane can be taken as the initially prestressed state and an admissible stress path is then determined accordingly. For instance, if the point a is the initially prestressed state, the path abde is an admissible stress path. So is the path agh for $\beta > 3$ and $\beta = 3$ but there is no corresponding path for $\beta < 3$ since the solid lines never intersect the σ_1-axis. If point d in Fig. 5(a) is the initially prestressed state, the stress paths dih and djm are both admissible. As a last example, suppose that the half-space is initially stressed beyond the initial yield surface as shown by the point i in Fig. 5(c), and at $t = 0$ a step load of σ_1 is added on the surface of the half-space such that the stress state changes to the point j in Fig. 5(c). The admissible stress path for this case is igj which shows that the shear stress will decrease first and increase later even though the shear stress on the surface of the half-space is kept constant.

Once we have an admissible stress path, a simple wave solution can be constructed following the examples shown in Figs. 3.

5. Concluding Remarks.

The problem of plane waves of combined pressure and shear stresses in an ideally elastic-plastic half-space originally studied by Bleich and Nelson is extended to a general elastic-plastic material. Simple wave solutions are obtained for the half-space which can be initially pre-stressed and the loads on the boundary need not be a step load. The simple wave solutions presented here of course, do not apply to

general boundary value problems. For general boundary value problems, a numerical scheme such as the method of characteristics using Eq. (25) must be used. The simple wave solutions obtained here, such as the one shown in Fig. 3(b), can be used as a test of the accuracy of the numerical scheme.

The existence of the fast waves and the slow waves has been verified experimentally by Lipkin and Clifton [10]. An example in which all the four wave speeds c_1, c_2, c_f and c_s are generated in one test is given in Fig. 6. For illustrative purpose, we assume the half-space is elastic-ideally plastic. The half-space is initially pre-stressed to the stress state indicated by point a in Fig. 6(a), and at time $t = 0$ the stress state at the boundary is changed to the stress sate indicated by point g in Fig. 6(a) and maintains at this constant state thereafter. For this case, the admissible stress path is abdeg. The portion ab is elastic, with wave speed c_1; the portion bd is plastic, with variable wave speeds c_f; the portion de is again elastic, with wave speed c_2; and the last portion eg is plastic, with variable wave speeds c_s. The corresponding simple wave solution is shown in Fig. 6(b).

In the wave propagation of combined stresses, an unexpected unloading may occur near the boundary when the stress state at the boundary suddenly changes from a lower yield surface to a higher yield surface, [2]. On the other hand, a plastic loading may occur when the stress state at the boundary suddenly changes from a higher yield stress to a lower yield stress, such as the path djm in Fig. 5(a). This phenomenon is not dictated by the fact that the stress state at the

boundary changes discontinuously. The same phenomenon exists if the stress state at the boundary changes continuously. In fact, it is shown in [11] that more than one elastic and plastic region can be generated near the boundary even though the stress state at the boundary is continuously changing from a lower yield surface to a higher yield surface and vice versa. This clearly indicates the hidden difficulties in solving wave propagation of combined stresses by a numerical scheme.

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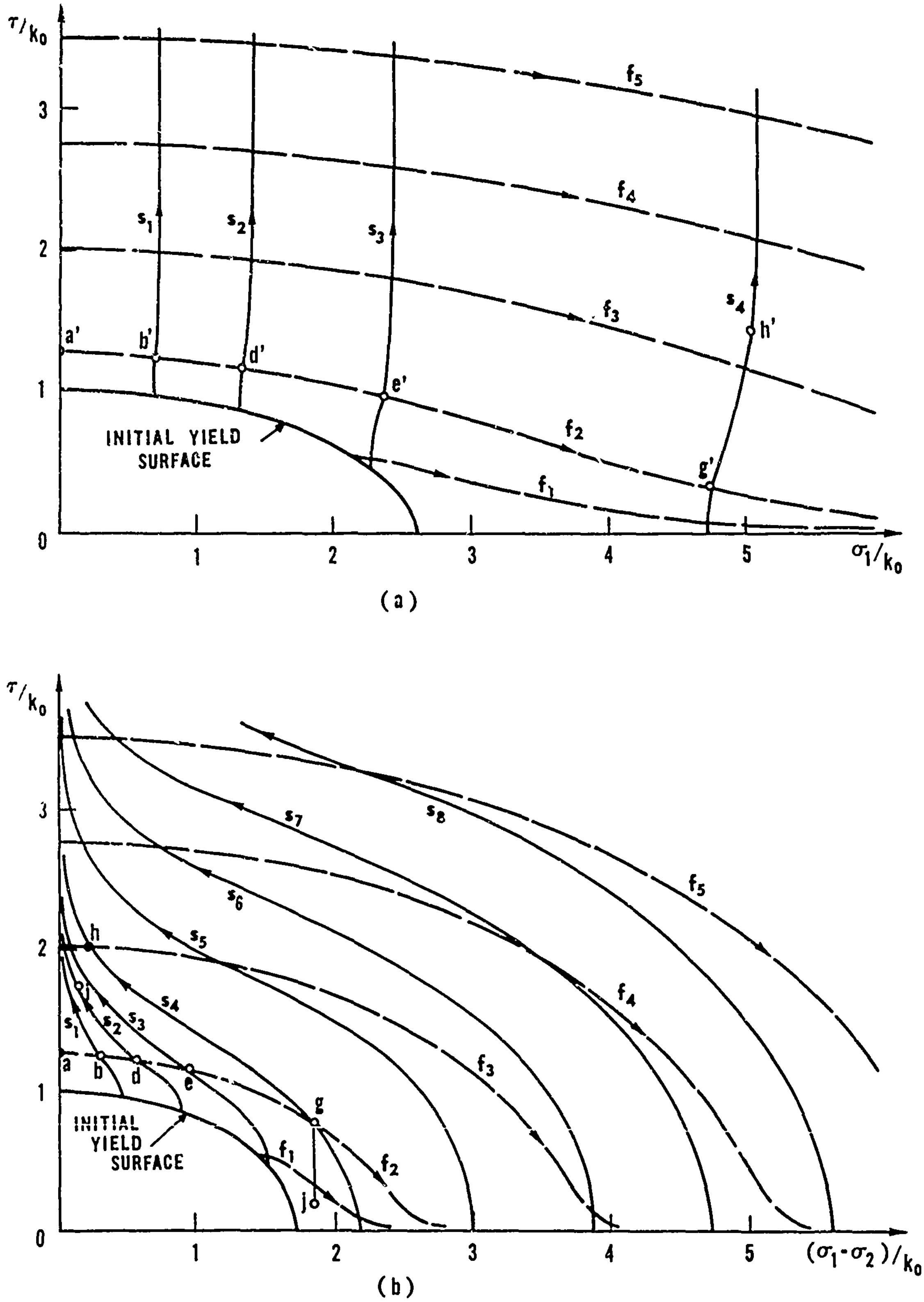


FIGURE 1. STRESS PATHS FOR SIMPLE WAVE SOLUTIONS
$\theta^2 = 3$, $\alpha = 10\,(k/k_o - 1)^{1/2}$, $\beta = 5$

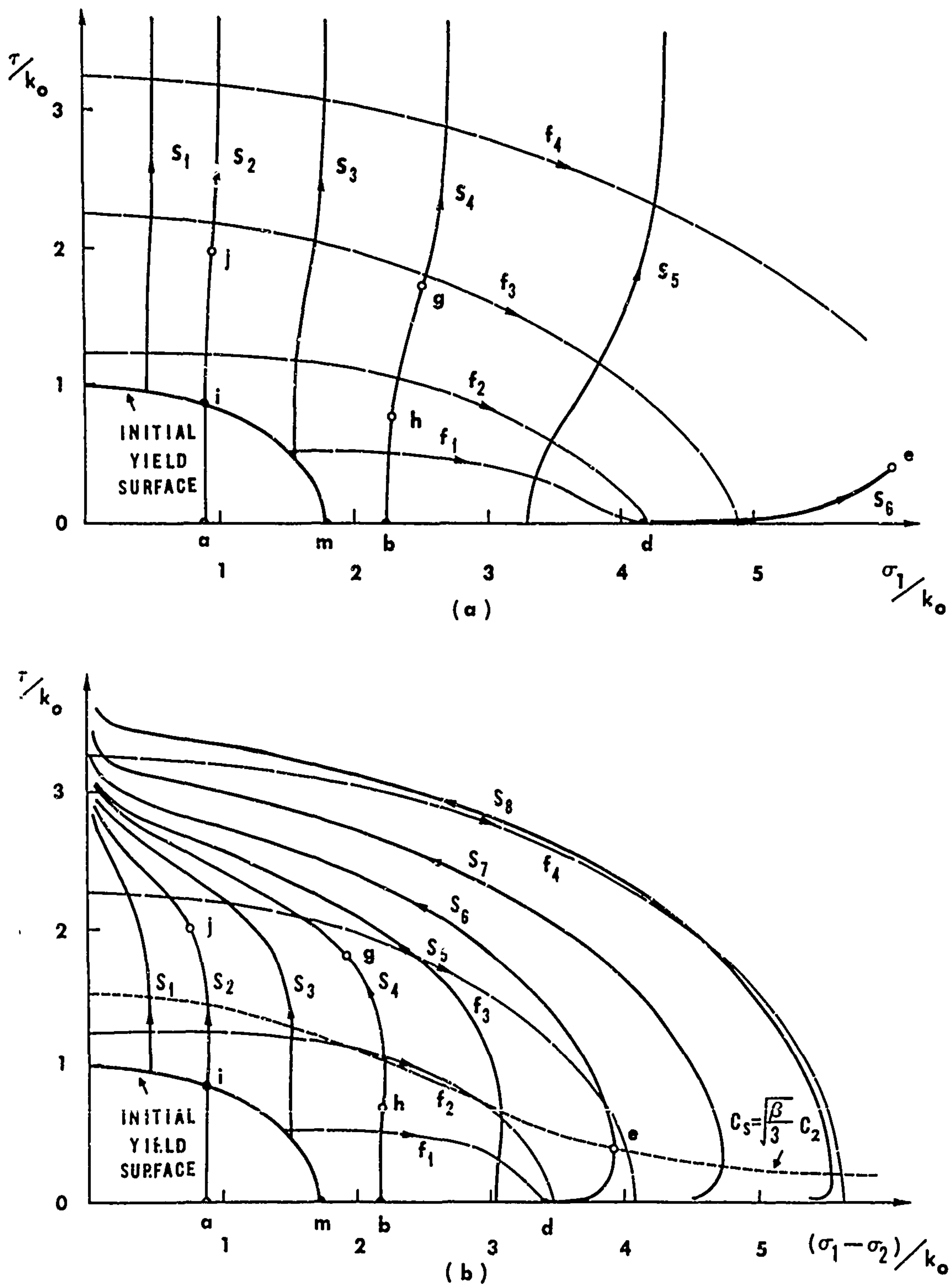
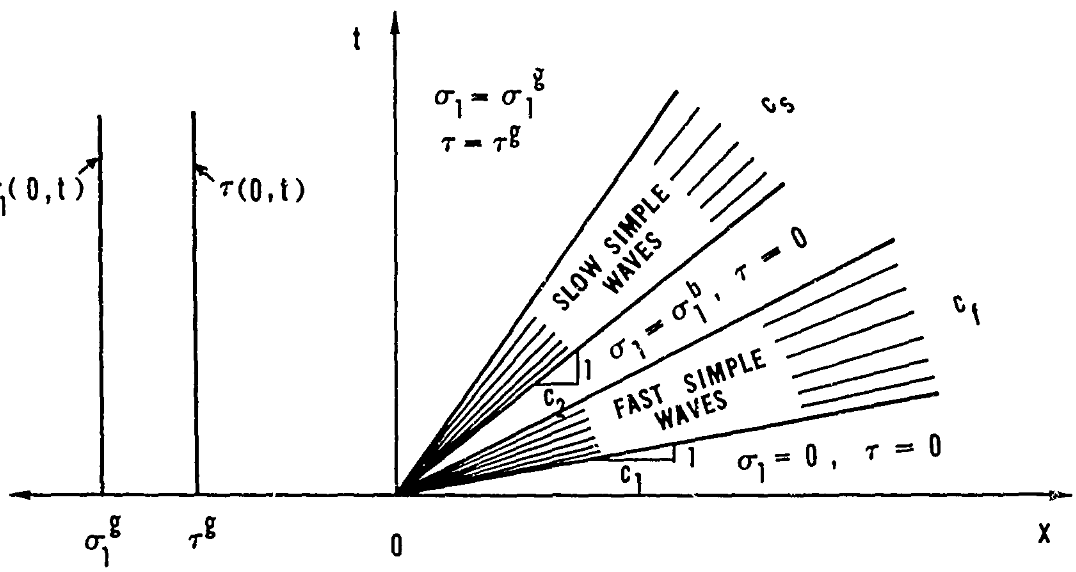
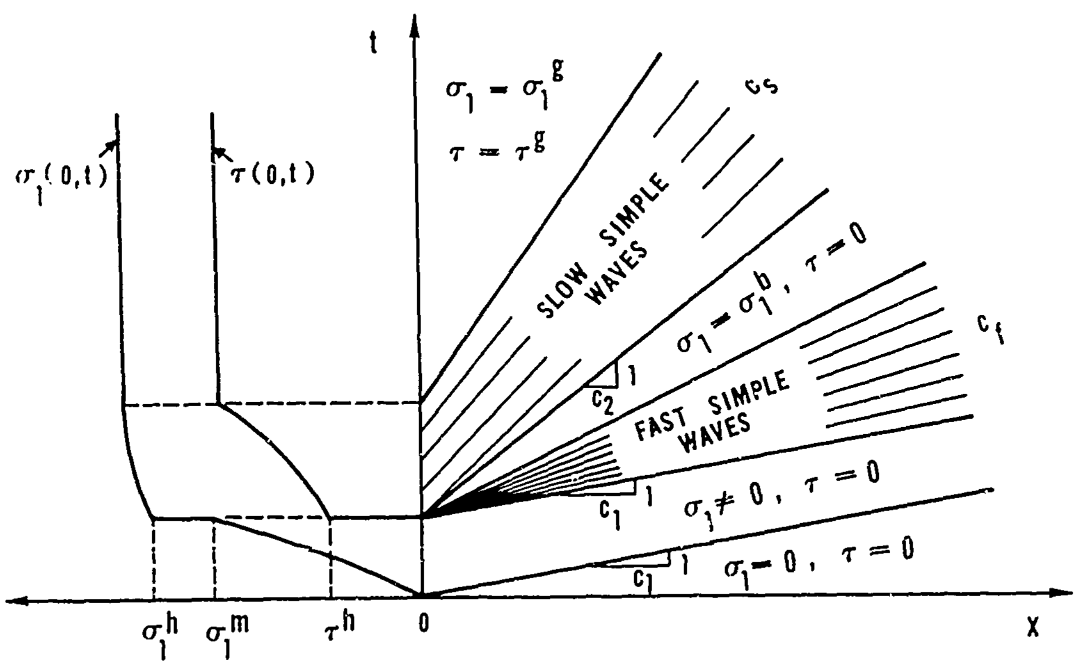


FIGURE 2. STRESS PATHS FOR SIMPLE WAVE SOLUTIONS
$\theta^2 = 3$, $\alpha = 3(k/k_o - 1)^4$, $\beta = 2.25$



(a) CENTERED SIMPLE WAVES



(b) GENERALIZED SIMPLE WAVES

FIGURE 3. SIMPLE WAVE SOLUTIONS FOR THE STRESS PATH obg OF FIGURE 2.

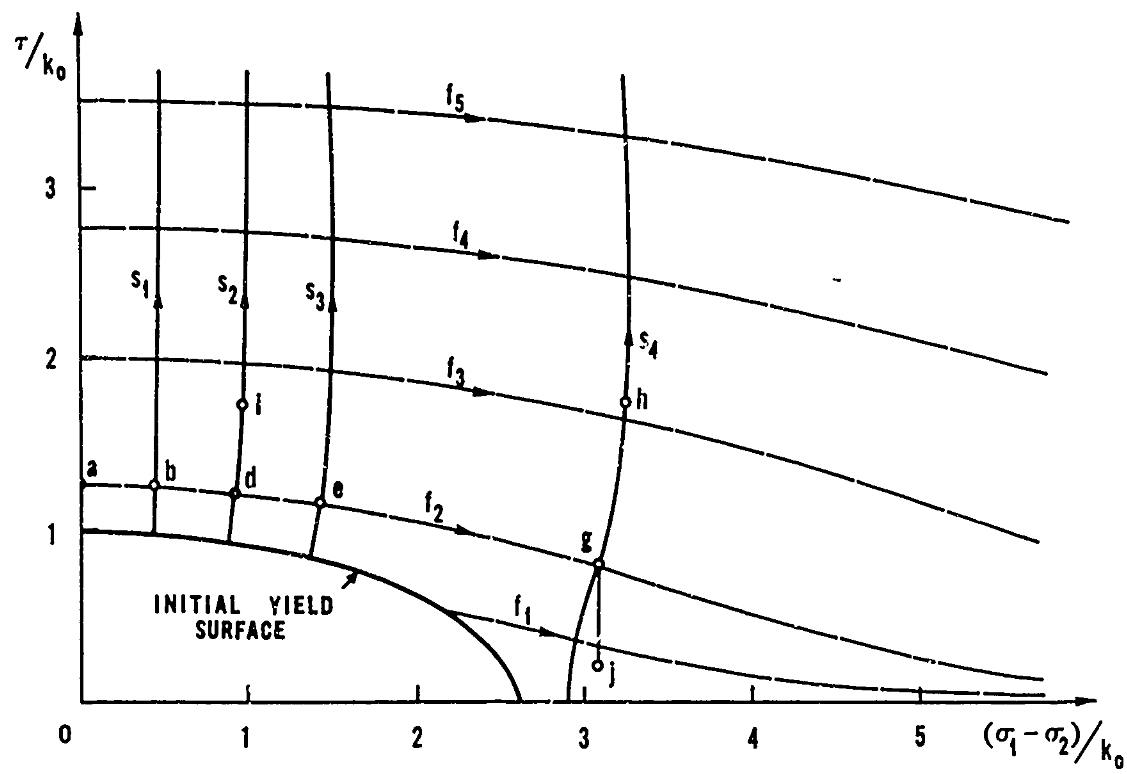


FIGURE 4. MODIFIED STRESS PATHS FOR FIGURE 1(a).

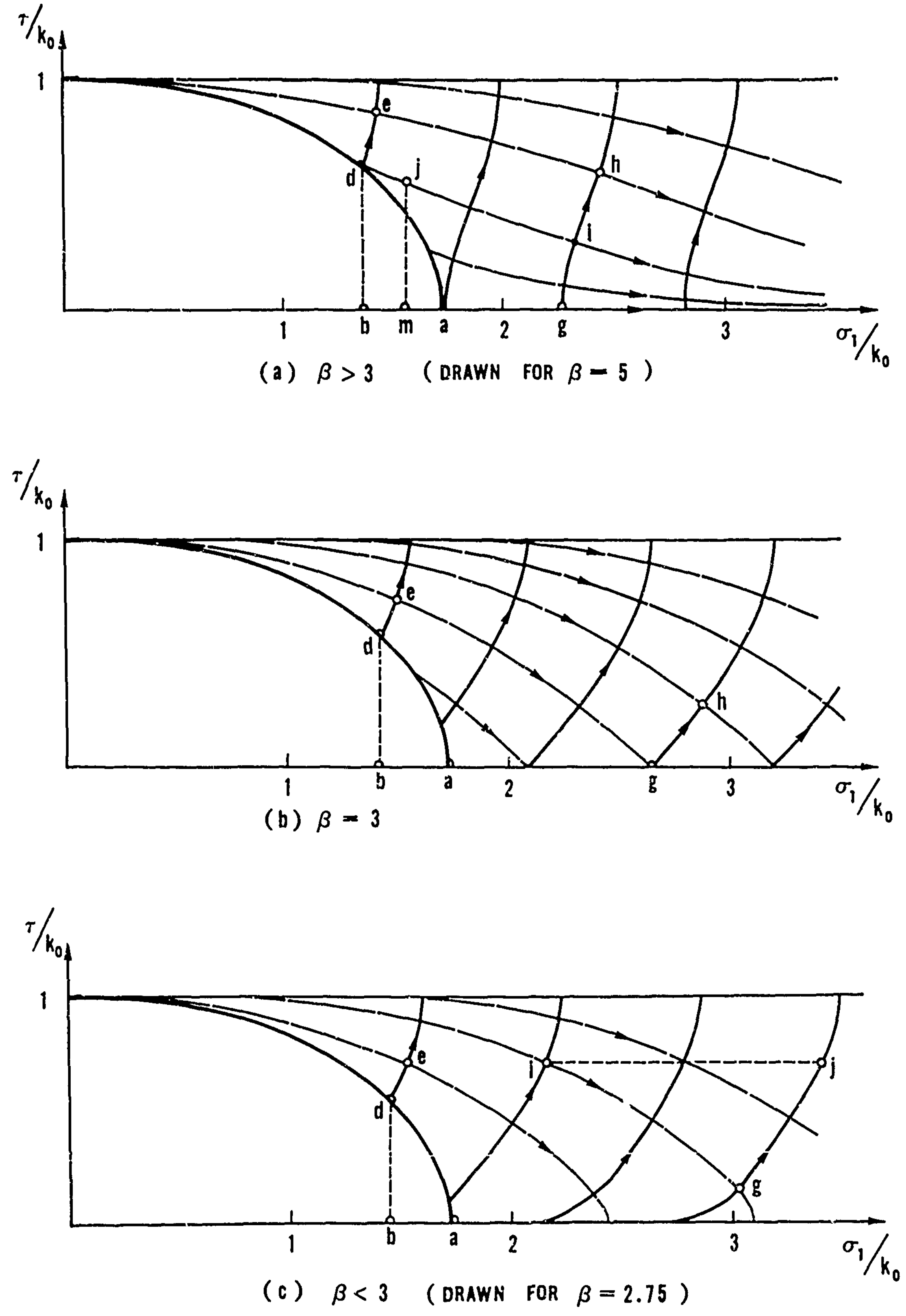
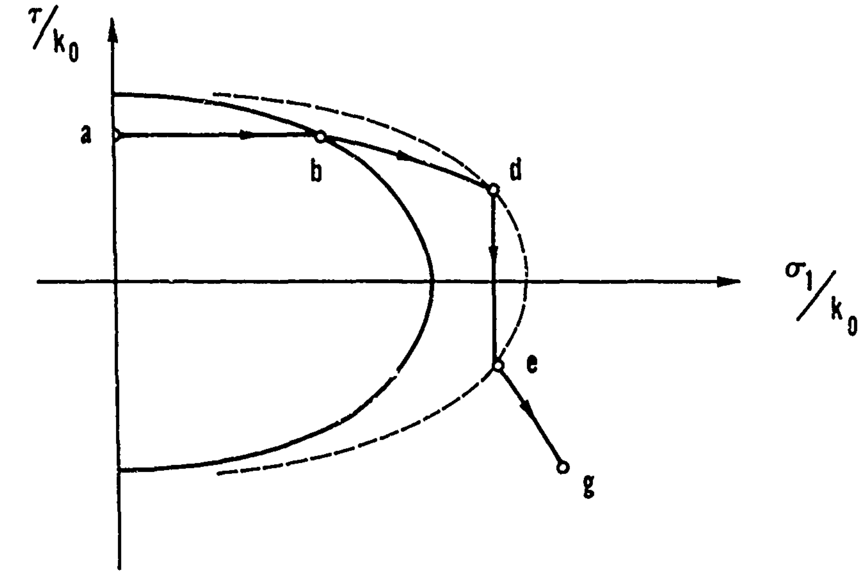
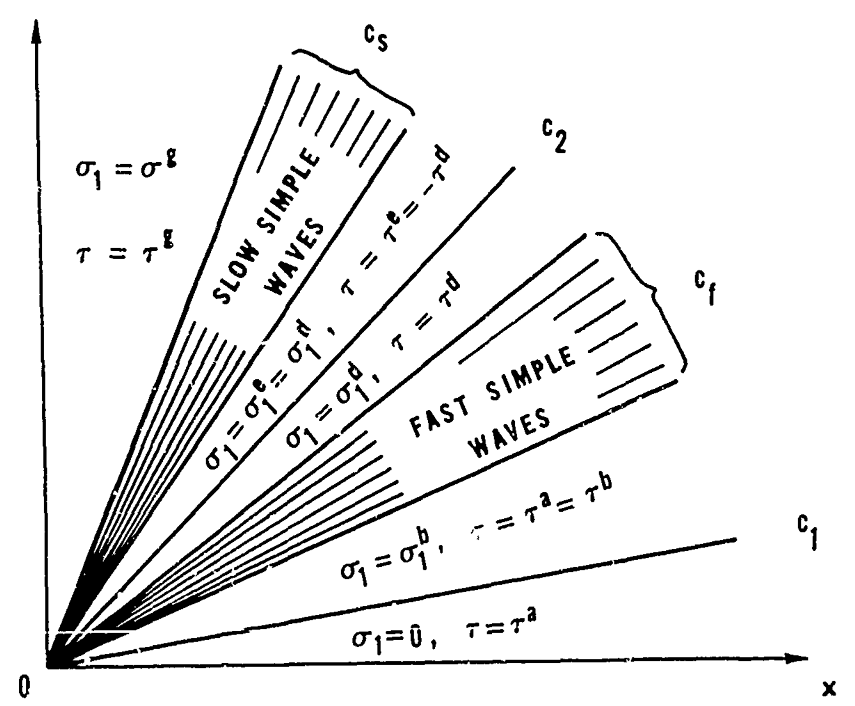


FIGURE 5. STRESS PATHS FOR SIMPLE WAVE SOLUTIONS FOR ELASTIC-PERFECTLY PLASTIC MATERIALS ($\theta^2 = 3$)



(a)



(b)

FIGURE 6. AN EXAMPLE OF CENTERED SIMPLE WAVES IN WHICH c_1, c_2, c_f, AND c_s ALL APPEAR.

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13. ABSTRACT

The plane wave propagation in a half-space due to a uniformly distributed step load of pressure and shear on the surface was first studied by Bleich and Nelson. The material in the half-space was assumed to be elastic-ideally plastic. In this paper, we study the same problem for a general elastic-plastic material. The half-space can be initially prestressed. The results can be extended to the case in which the loads on the surface are not necessarily step loads, but with a restricted relation between the pressure and the shear stresses.

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